Bulk induced boundary perturbations for $\mathcal{N} = 1$ superconformal field theories

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Abstract

The $\mathcal{N} = 1$ superconformal circle theory consisting of a free boson and a free fermion is considered. At any radius the theory has standard Dirichlet and Neumann branes, but for rational radii there are additional superconformal boundary conditions that are labelled by elements in a quotient of SU(2). We analyse how these branes behave under the radius-changing bulk perturbation. As in the bosonic case, the bulk perturbation induces in general a boundary RG flow whose end-point is a superposition of Dirichlet or Neumann branes.

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1 Introduction

Many closed string backgrounds possess moduli that allow one to change the shape and size of the background geometry. Furthermore, the possible D-branes of a given background also typically form a moduli space [1]. Obviously, these two moduli spaces are not unrelated: the moduli space of D-branes typically depends on where one sits in the closed string moduli space, and conversely, D-branes backreact on the geometry and may have an impact on whether some of the closed string moduli may get lifted. It is clearly an important question to understand in some detail how these two moduli spaces are related to one another.

Recently, some progress has been made by studying this question from a conformal field theory point of view. As was shown in [2, 3], an exactly marginal bulk operator (describing a bulk deformation) can cease to be exactly marginal in the presence of a boundary. If this is the case, it induces a non-trivial RG flow on the boundary that drives the boundary condition to one that is compatible with the deformed closed string background.

As an example, this process was studied for the case of a single free boson in [2]. The moduli space of D-branes for this theory depends crucially on the radius of the circle [4, 5, 6]: for all radii there are Neumann and Dirichlet branes, but if the radius is a rational multiple of the self-dual radius, there is an additional 3-dimensional branch of the moduli space of conformal D-branes. On the other hand, for irrational multiples of the self-dual radius, the additional branch of the moduli space is only 1-dimensional. The structure of the full moduli space of conformal D-branes thus changes very discontinuously as one varies the radius of the circle theory. In fact, if one starts with a generic brane at a rational point, then the radius-changing bulk perturbation is not exactly marginal, but rather induces a non-trivial RG flow on the boundary. In the example at hand this RG flow could be solved exactly (using the equivalence of the theory at the self-dual radius to the SU(2) WZW model at $k = 1$), and the end-point of the flow could be determined [2]: if the radius is increased, a generic brane always flows to a (superposition of) Dirichlet branes, while if the radius is decreased, the endpoint of the flow is a (superposition of) Neumann branes.

In this paper we study the $\mathcal{N} = 1$ supersymmetric analogue of this problem. The moduli space of $\mathcal{N} = 1$ superconformal branes for the free boson and fermion theory has a similar structure as in the bosonic case [4, 7], and there is also a close relation to the SU(2) WZW model, this time at $k = 2$ [8, 9].[1] However, there are also some differences: the WZW model description only applies to the superaffine theory at $R = 1$ (not one of the circle theories), and one needs to keep track carefully of the GSO-projection. As we shall show, one can overcome these difficulties and obtain as complete a picture as in the bosonic example. In particular, one finds that generic branes flow to superpositions of Dirichlet or Neumann branes as the radius of the circle is increased or decreased, respectively.

Bulk induced boundary perturbations have also been discussed in [11, 12, 13, 14], as well as in the context of defect operators [15, 16]. The backreaction effect has also been analysed from this point of view in [17].

The paper is organised as follows. In section 2 we briefly review the salient features

[1] The rational boundary states for all multicritical points were also constructed in [10].

of the bosonic analysis. Section 3 explains how the $\mathcal{N}=1$ free boson and free fermion theory is related to the WZW model at $k=2$, and the corresponding boundary states are identified in section 4. In section 5 we then put everything together and deduce the RG flow of the $\mathcal{N}=1$ superconformal branes from the WZW analysis. Section 6 contains our conclusions. There is one appendix containing some technical calculation.

2 Review of the bosonic analysis

Let us begin by reviewing briefly the analysis in the bosonic case. We consider the $c=1$ conformal field theory of a single free boson compactified at a radius R. At the self-dual radius $R=1/\sqrt{2}$ — in our conventions $\alpha'=\frac{1}{2}$ — the theory is equivalent to the SU(2) WZW model at level 1, where the left-moving currents are expressed in terms of the left-moving free boson field X_L as

$$J^3(z) \ := \ i\sqrt{2}\,\partial_z X_L(z)\ , \qquad J^{\pm}(z) \ := :\exp\big(\pm 2\sqrt{2} i\, X_L(z)\big):\ , \tag{2.1}$$

and similarly for the right-movers such as $\bar{J}^3=-i\sqrt{2}\partial_{\bar{z}}X_R$. It was shown in [18] that the full moduli space of conformal boundary conditions for this theory is precisely the group manifold SU(2). The corresponding boundary conditions preserve the su(2) affine symmetry up to conjugation by a group element; we choose the conventions that the boundary condition labelled by g satisfies the gluing condition

$$\Big(\mathrm{Ad}_{(g\cdot\iota)}(J^a_n) \ + \ \bar{J}^a_{-n}\Big)\,\|g\rangle\!\rangle_{\mathrm{WZW}} \ = \ 0\ , \tag{2.2}$$

where ι is the SU(2) matrix $\iota:=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, and Ad denotes the adjoint representation of SU(2). In these conventions, a diagonal group element describes a Dirichlet brane, while off-diagonal group elements correspond to Neumann branes.

If the radius of the circle is a rational multiple of the self-dual radius, $R=\frac{M}{N}\frac{1}{\sqrt{2}}$, then the conformal boundary conditions are labelled by elements in the quotient space $\mathrm{SU}(2)/\mathbb{Z}_M\times\mathbb{Z}_N$. In addition, there are also the usual Dirichlet and Neumann branes (that exist at any radius). On the other hand, if the radius is an *irrational* multiple of the self-dual radius, the moduli space of boundary conditions is much smaller [4, 5, 6].

In the bulk theory, the radius R is a modulus of the theory, *i.e.* it corresponds to an exactly marginal bulk operator Φ. Changing the radius then corresponds to the perturbation of the action by

$$\mathcal{S}_{\delta R} \ = \ \lambda\int \mathrm{d}^2z\ \Phi(z,\bar{z}) = \ \lambda\int \mathrm{d}^2z\ J^3(z)\,\bar{J}^3(\bar{z}) \ = \ 2\,\lambda\int \mathrm{d}^2z\ \partial_z X\,\partial_{\bar{z}}X\ , \tag{2.3}$$

where in our conventions $\lambda>0$ means that the radius R is decreased. As we have mentioned above, the moduli space of D-branes depends in a very discontinuous manner on the radius, and thus the effect of this bulk perturbation on the boundary conditions must be non-trivial. This question was studied in [2], where it was shown that the above bulk perturbation is not necessarily exactly marginal in the presence of a boundary, but rather induces in general a non-trivial RG flow on the boundary. If we denote the boundary coupling constants by μ_k, then the relevant RG equations are of the form

$$\dot{\mu}_k \ = \ (1-h_k)\,\mu_k \ + \ \frac{1}{2}\,B_{\Phi k}\,\lambda \ + \ D_{ijk}\,\mu_i\,\mu_j \ + \ \mathcal{O}(\mu\lambda,\mu^3,\lambda^2)\ . \tag{2.4}$$

Here h_k is the conformal dimension of the boundary field corresponding to μ_k — in our case the boundary fields of interest are just the currents whose conformal dimension is equal to 1 — while $B_{\Phi k}$ denotes the bulk-boundary coupling constant of the perturbing bulk field Φ, and D_{ijk} are the boundary OPE coefficients. For the case at hand, the bulk boundary coefficient could be determined explicitly as[2]

$$B_{\Phi\gamma} \;=\; i\,\mathrm{Tr}\Big(t^\gamma\,[t^3, g t^3 g^{-1}]\Big)\;, \tag{2.5}$$

where t^α are the generators of the Lie algebra su(2), and γ labels the boundary current J^γ, while the group element g characterises the boundary condition in question. To first order in the bulk perturbation, the induced boundary RG flow only changes the group element g, and it is then possible to integrate up the induced boundary flow completely (to first order in λ). If we label the group elements in SU(2) as

$$g \;=\; \begin{pmatrix} e^{i\phi}\,\cos\theta & ie^{i\psi}\,\sin\theta \\ ie^{-i\psi}\,\sin\theta & e^{-i\phi}\,\cos\theta \end{pmatrix}\;, \tag{2.6}$$

then the brane flow only affects θ; if the radius is increased (decreased) the brane labelled by g flows to a pure Dirichlet (Neumann) brane whose value of ϕ (ψ) is unchanged.

3 Equivalence of bulk theories

In the following we want to repeat this analysis for the $\mathcal{N}=1$ superconformal field theory consisting of a free boson and a free fermion. As we have seen above, the bosonic analysis was simplest in the WZW description of the theory. In the superconformal case, there is also a direct relation to a WZW model — this time the su(2) WZW model at level 2 — and it will again be convenient to use this formulation of the theory. In the superconformal context the WZW model is not directly equivalent to the theory of a free boson and fermion, but rather to the so-called superaffine theory [8].

Let us first describe the WZW model in some detail. It is well known that the su(2) level 2 theory has a free field realisation in terms of three free Majorana fermions, see for example [19]. Let us denote the corresponding fermion fields as $\psi^a(z)$, where a denotes the adjoint representation of su(2), and we have the commutation relations

$$\{\psi^a_r, \psi^b_s\} \;=\; \delta_{r,-s}\,\delta^{ab}\;. \tag{3.1}$$

Then the WZW currents are given as

$$J^a(z) \;=\; -\frac{i}{2}\,\varepsilon^{abc}\;:\psi^b(z)\,\psi^c(z): \quad\Longleftrightarrow\quad J^a_n \;=\; -\frac{i}{2}\sum_r \varepsilon^{abc}\;:\psi^b_{n-r}\,\psi^c_r:\;, \tag{3.2}$$

where ε^{abc} is the totally antisymmetric tensor in three dimensions.

At level 2, the possible representations of the su(2) affine algebra are $\mathcal{H}_j$ with $j=0,\frac{1}{2},1$, and the complete space of states is of the form

$$\mathcal{H}_{\rm WZW} \;=\; \big(\mathcal{H}_0\otimes\bar{\mathcal{H}}_0\big)\;\oplus\;\big(\mathcal{H}_{\frac{1}{2}}\otimes\bar{\mathcal{H}}_{\frac{1}{2}}\big)\;\oplus\;\big(\mathcal{H}_1\otimes\bar{\mathcal{H}}_1\big)\;. \tag{3.3}$$

[2]Because the boundary conditions are labelled with the inclusion of ι, there is now a different sign compared to [2]. Note however that $\lambda>0$ now corresponds to decreasing the radius.

In terms of the free fermion description, the first and last term arise from the NS-NS sector, while the middle term (the representation $j=\frac{1}{2}$) corresponds to the R-R sector. Each of these sectors is moded out by the GSO-projection $\frac{1}{2}(1+(-1)^{F+\tilde{F}})$; in fact, we have the simple relation between the $\hat{su}(2)_2$ characters

$$\chi_{j=0}(q) \;=\; \frac{f_3(q)^3+f_4(q)^3}{2}\,, \quad \chi_{j=1}(q) \;=\; \frac{f_3(q)^3-f_4(q)^3}{2}\,, \quad \chi_{j=\frac{1}{2}}(q) \;=\; \frac{1}{\sqrt{2}} f_2(q)^3\,, \tag{3.4}$$

where the functions $f_i(q)$ are the usual functions of [20] that describe the characters of free fermion representations, see (A.2).

This WZW model is now equivalent to the so-called superaffine theory that is defined as a $\mathbb{Z}_2$ orbifold of the free boson and fermion circle theory at radius $R=1$. Let us denote the modes of the boson field X of the circle theory by α_m and $\bar{\alpha}_m$, while the modes of the free fermion fields $\chi(z)$ and $\bar{\chi}(\bar{z})$ are denoted by χ_r and $\bar{\chi}_r$; the commutation relations are

$$[\alpha_m,\alpha_n] \;=\; m\,\delta_{m,-n}\,, \qquad [\alpha_m,\chi_r] \;=\; 0\,, \qquad \{\chi_r,\chi_s\} \;=\; \delta_{r,-s} \tag{3.5}$$

and similarly for the right-moving modes. The orbifold acts as

$$\mathcal{S} \;=\; \Big(X\mapsto X+\pi R\Big)\times(-1)^{F_{st}}\,, \tag{3.6}$$

and F_{st} denotes the left-moving spacetime fermion number, such that $(-1)^{F_{st}}$ acts as $+1$ (-1) on the NS-NS (R-R) sector. In the untwisted sector of this orbifold, the left- and right-moving momenta are thus of the form

$$\text{untwisted:}\quad (p_L,p_R) \;=\; \left(\frac{k}{2}+w\,,\;\frac{k}{2}-w\right)\,, \qquad \left\{\begin{array}{rl} k\in 2\mathbb{Z},\ w\in\mathbb{Z} & :\ \text{NSNS} \\ k\in 2\mathbb{Z}-1,\ w\in\mathbb{Z} & :\ \text{RR}\,, \end{array}\right. \tag{3.7}$$

while in the twisted sector we have instead

$$\text{twisted:}\quad (p_L,p_R) \;=\; \left(\frac{k}{2}+w\,,\;\frac{k}{2}-w\right)\,, \qquad \left\{\begin{array}{rl} k\in 2\mathbb{Z}-1,\ w\in\mathbb{Z}-\frac{1}{2} & :\ \text{NSNS} \\ k\in 2\mathbb{Z},\ w\in\mathbb{Z}-\frac{1}{2} & :\ \text{RR}\,. \end{array}\right. \tag{3.8}$$

In the twisted sector the GSO-projection is reversed; thus in the twisted NS-NS sector the momentum ground state is now odd under the GSO-projection.

It is instructive to understand how the currents of the WZW description appear in the superaffine orbifold. First we observe that the currents ∂X and $\bar{\partial}X$ are invariant under the orbifold projection; they correspond to the two currents J^3 and $\bar{J}^3$ in the WZW language. The other currents of the WZW theory arise as the fermionic descendants of the momentum ground states of the twisted sector

$$J^{\pm} \quad\Longleftrightarrow\quad \chi_{-1/2}\,|(p_L=\pm 1,p_R=0)\rangle\,, \qquad \bar{J}^{\pm} \quad\Longleftrightarrow\quad \bar{\chi}_{-1/2}\,|(p_L=0,p_R=\pm 1)\rangle\,. \tag{3.9}$$

It is also fairly straightforward to show that the partition function of the WZW model agrees with that of the superaffine theory. This is most easily seen by writing the WZW model partition function in terms of free fermion characters

$$Z_{\text{WZW}}(q,\bar{q}) \;=\; \frac{1}{2}\Big(|f_3(q)|^6+|f_4(q)|^6+|f_2(q)|^6\Big)\,, \tag{3.10}$$

as follows from (3.4). On the other hand, using the sum representations of the theta functions, one can show that (3.10) agrees with the partition function coming from the momentum lattice (3.7) and (3.8).

Finally, we note that we can also obtain the circle theory from the superaffine theory by doing the 'quantum orbifold'. In the present case this is the winding shift orbifold

$$\tilde{\mathcal{S}} = \left(\tilde{X} \mapsto \tilde{X} + \frac{\pi}{R}\right) , \tag{3.11}$$

where $\tilde{X}$ is the dual coordinate, *i.e.* $\tilde{X} = X_L - X_R$, see also [9].

4 Boundary conditions

In order to analyse the bulk induced boundary flow for the $\mathcal{N} = 1$ circle theory, we shall proceed in two steps. We shall first analyse the situation for the superaffine theory (which is equivalent to the WZW model at level 2), and then deduce from this the results for the circle theory at radius $R = 1$ by considering the circle theory as the $\mathbb{Z}_2$ orbifold of the superaffine theory. In order to translate between the different descriptions, we first need to understand the dictionary between the brane descriptions in the different setups in some detail; some aspects of this were already analysed in [9].

4.1 Branes in the superaffine theory

We first review the description of the superaffine branes from [9]. We are interested in the branes $\|B\rangle\!\rangle$ that preserve the superconformal (but not necessarily any larger) symmetry. The corresponding gluing conditions read

$$\left(L_n \ - \ \bar{L}_{-n}\right) \|B\rangle\!\rangle \ = \ 0 \ = \ \left(G_r \ + \ i\eta\, \bar{G}_{-r}\right) \|B\rangle\!\rangle \ , \tag{4.1}$$

where G (and $\bar{G}$) denotes the supercurrent of the $\mathcal{N} = 1$ superconformal algebra — we use the same conventions as in [4] — and $\eta = \pm$ labels the two possible choices for the $\mathcal{N} = 1$ gluing conditions.

As explained in [9], the Ishibashi states of the superaffine theory at $R = 1$ are labelled by the usual triplets $(j; m, n)$, together with the sign $\eta = \pm$. In the NS-NS sector all combinations with j integer appear, while j is half-integer in the R-R sector. We also choose the convention (as in [9]) that the R-R sector Ishibashi states are only GSO-invariant for $\eta = -$. With these preparations we can then give an explicit formula for the boundary states of the superaffine theory. Depending on the choice of η we have

$$\begin{aligned} \|g; -\rangle\!\rangle_{sa} \ &= \ \frac{1}{\sqrt{2}} \left(\sum_{(j,m,n) \in \mathcal{I}^{\mathrm{NS}}} D^j_{m,n}(g)\, |j; m, n; -\rangle\!\rangle^{\mathrm{NS}} \ + \sum_{(j,m,n) \in \mathcal{I}^{\mathrm{R}}} D^j_{m,n}(g)\, |j; m, n; -\rangle\!\rangle^{\mathrm{R}} \right) \\ \|g; +\rangle\!\rangle_{sa} \ &= \ \sum_{(j,m,n) \in \mathcal{I}^{\mathrm{NS}}} D^j_{m,n}(g)\, |j; m, n; +\rangle\!\rangle^{\mathrm{NS}} \ . \end{aligned} \tag{4.2}$$

Here $\mathcal{I}^{\mathrm{NS}}$ contains all the integer spin triplets $(j; m, n)$, whereas $\mathcal{I}^{\mathrm{R}}$ covers all the half-integer spin cases.

We shall often refer to the first family of branes ($\eta = -$) as the BPS branes, while the second family ($\eta = +$) will be called non-BPS. It is easy to see that the moduli space of the BPS branes is precisely SU(2), *i.e.* branes corresponding to different group elements are indeed different. On the other hand, for $\eta = +$, the boundary states corresponding to g and $-g$ are identical, and thus the moduli space is in fact SO(3)=SU(2)/$\mathbb{Z}_2$.

In general these boundary states only preserve the superconformal symmetry, but there are special cases that actually preserve more. In particular, one easily checks that

$$\begin{aligned}\Big(\alpha_n \;-\; \bar\alpha_{-n}\Big) \,\| \left(\begin{smallmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{smallmatrix}\right); \eta\rangle\!\rangle_{sa} \;&=\; \Big(\chi_r \;+\; i\eta\,\bar\chi_{-r}\Big) \,\| \left(\begin{smallmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{smallmatrix}\right); \eta\rangle\!\rangle_{sa} \;=\; 0 \\ \Big(\alpha_n \;+\; \bar\alpha_{-n}\Big) \,\| \left(\begin{smallmatrix} 0 & ie^{i\psi} \\ ie^{-i\psi} & 0\end{smallmatrix}\right); \eta\rangle\!\rangle_{sa} \;&=\; \Big(\chi_r \;-\; i\eta\,\bar\chi_{-r}\Big) \,\| \left(\begin{smallmatrix} 0 & ie^{i\psi} \\ ie^{-i\psi} & 0\end{smallmatrix}\right); \eta\rangle\!\rangle_{sa} \;=\; 0\,. \end{aligned} \tag{4.3}$$

The branes associated to off-diagonal group elements are single Neumann branes, while for diagonal group elements they always describe a superposition of two Dirichlet branes at opposite points on the circle; in the BPS case ($\eta = -$), the two Dirichlet branes are a brane-anti-brane pair, while in the non-BPS case ($\eta = +$) the two Dirichlet branes are both non-BPS branes. From the point of view of the superaffine theory, both of these configurations are however fundamental, *i.e.* cannot be resolved into more elementary branes.

As we mentioned before, the superaffine theory is the $\mathbb{Z}_2$ orbifold of the circle theory, and vice versa. Given the branes of either theory, we can obtain the branes of the other theory by the usual orbifold construction. This was explained in detail in [9].

4.2 The WZW description

As we shall now explain, all of these branes correspond to D-branes of the WZW model that preserve the affine symmetry up to conjugation, *i.e.* that satisfy

$$\Big(\mathrm{Ad}_{(g\cdot\iota)}(J^a_n) \;+\; \bar J^a_{-n}\Big)\,\|g\rangle\!\rangle_{\rm WZW} \;=\; 0\,. \tag{4.4}$$

It is straightforward to construct the corresponding boundary states following [21]. The relevant Ishibashi states [22] are simply obtained from the usual ($g = \iota^{-1}$) Ishibashi states by the action of $(g\cdot\iota)$; thus we have three families of Ishibashi states $|g;j\rangle\!\rangle$, coming from the three different sectors $j = 0, \frac{1}{2}, 1$ of the theory. Expressed in terms of the Ishibashi states of the superaffine theory labelled by (j,m,n), the relation is

$$\begin{aligned} |g;0\rangle\!\rangle \;+\; |g;1\rangle\!\rangle \;&=\; \sum_{(j,m,n)\in\mathcal{I}^{\rm NS}} D^j_{m,n}(g)\,|j;m,n;-\rangle\!\rangle^{\rm NS} \\ |g;0\rangle\!\rangle \;-\; |g;1\rangle\!\rangle \;&=\; \sum_{(j,m,n)\in\mathcal{I}^{\rm NS}} D^j_{m,n}(g)\,|j;m,n;+\rangle\!\rangle^{\rm NS} \\ |g;\tfrac{1}{2}\rangle\!\rangle \;&=\; 2^{-\frac{1}{4}} \sum_{(j,m,n)\in\mathcal{I}^{\rm R}} D^j_{m,n}(g)\,|j;m,n;-\rangle\!\rangle^{\rm R}\,. \end{aligned} \tag{4.5}$$

Given the description of the Ishibashi states, it is then straightforward to construct boundary states following [21]. Using the explicit form of the S-matrix for su(2) at level 2,

$$\mathcal{S}_{j,j'} \;=\; \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}\,, \tag{4.6}$$

the consistent boundary states are

$$\begin{aligned}
\|g;0\rangle\!\rangle_{\rm WZW} &= \frac{1}{\sqrt{2}}\Big(|g;0\rangle\!\rangle + |g;1\rangle\!\rangle\Big) + \frac{1}{2^{\frac{1}{4}}}\,|g;\tfrac{1}{2}\rangle\!\rangle = \|g;-\rangle\!\rangle_{sa} \\
\|g;\tfrac{1}{2}\rangle\!\rangle_{\rm WZW} &= |g;0\rangle\!\rangle - |g;1\rangle\!\rangle = \|g;+\rangle\!\rangle_{sa} \\
\|g;1\rangle\!\rangle_{\rm WZW} &= \frac{1}{\sqrt{2}}\Big(|g;0\rangle\!\rangle + |g;1\rangle\!\rangle\Big) - \frac{1}{2^{\frac{1}{4}}}\,|g;\tfrac{1}{2}\rangle\!\rangle = \|-g;-\rangle\!\rangle_{sa}\,.
\end{aligned} \tag{4.7}$$

Since we can express the WZW Ishibashi states in terms of the Ishibashi states of the superaffine theory (4.5), we can then also deduce the identification of the WZW model boundary states with those of the superaffine theory (see (4.7)). Note that the first and last line are compatible, since one knows on general grounds (see for example [2]) that

$$\|g;j\rangle\!\rangle_{\rm WZW} = \|-g;\tfrac{k}{2}-j\rangle\!\rangle_{\rm WZW}\,, \tag{4.8}$$

in agreement with the identification in terms of superaffine branes. Applied to $j=\frac{1}{2}$, this observation also implies that the branes in the middle line are only associated to SO(3)=SU(2)/$\mathbb{Z}_2$. One can also verify that this identification in agreement with the various cylinder overlaps; this is explained in more detail in appendix A.

Finally, we mention in passing that these branes also have a simple description in terms of the free fermionic description of the WZW model: the relevant gluing conditions are of the form

$$\Big(\mathrm{Ad}_{(g\cdot\iota)}(\psi^a_r) + i\eta\,\bar{\psi}^a_{-r}\Big)\,\|g;\eta\rangle\!\rangle_\psi = 0\,. \tag{4.9}$$

However, this will not be important for the rest of our analysis.

5 The boundary flow

Now we are ready to analyse the boundary flow in these theories. We begin by studying the branes of the superaffine theory. We are interested in the perturbation by the bulk field

$$\Phi = J^3\,\bar{J}^3 = 4\,\partial_z X\,\bar{\partial}_{\bar{z}} X\,, \tag{5.1}$$

where the factor of 4 takes into account that the currents are differently normalised at $k=2$. As we have explained above in section 3, J^3 corresponds to the current ∂X of the superaffine theory that survives the orbifold from the circle theory. Thus Φ describes indeed the radius-changing modulus we are interested in. Note also that this field is the $G_{-1/2}\bar{G}_{-1/2}$ descendant of the $h=\bar{h}=\frac{1}{2}$ field $\chi\bar{\chi}$ and hence preserves the $\mathcal{N}=1$ superconformal symmetry in the bulk.

5.1 The superaffine case

In terms of the WZW model, the analysis is essentially identical to what was done in [2] and reviewed in section 2. In fact, the level of the WZW model only enters in a rather trivial way, namely as an overall factor in front of (2.5), and hence the calculation and the conclusions are exactly as described there. In terms of the superaffine boundary states, this then implies that

$$\|\left(\begin{smallmatrix} 0 & ie^{i\psi} \\ ie^{-i\psi} & 0\end{smallmatrix}\right);\eta\rangle\!\rangle_{sa} \quad \overset{\delta R<0}{\Longleftarrow} \quad \|g;\eta\rangle\!\rangle_{sa} \quad \overset{\delta R>0}{\Longrightarrow} \quad \|\left(\begin{smallmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{smallmatrix}\right);\eta\rangle\!\rangle_{sa}\,. \tag{5.2}$$

As was explained in section 4.1, the diagonal and off-diagonal boundary conditions correspond to Dirichlet and Neumann branes, whose position and Wilson line is determined by the phase of the unperturbed SU(2) element. Thus we conclude that the superaffine branes flow to (a superposition of) Dirichlet or Neumann branes as the radius is increased or decreased, respectively.[3] This mirrors precisely the result obtained in [2] for the bosonic $c = 1$ theory. This conclusion is independent of whether these branes are BPS or non-BPS.

5.2 The circle theory at $R = 1$

Having understood the boundary flow for the superaffine theory, we can now use the fact that the circle theory at radius $R = 1$ is the $\mathbb{Z}_2$ orbifold of the superaffine theory, to deduce what happens in the circle theory. The quantum symmetry by means of which we can obtain the circle theory from the superaffine theory was already given in (3.11). By construction, this orbifold projects out the twisted sector of the original $\mathcal{S}$-orbifold; in particular, it removes all states of half-integer winding from the spectrum (see (3.8)). On the superaffine branes, the orbifold acts as $\tilde{\mathcal{S}}\, \|g;\eta\rangle\rangle_{sa} = \|\sigma_3 g \sigma_3;\eta\rangle\rangle_{sa}$; the orbifold invariant boundary states (that define the boundary states of the circle theory) are then [9]

$$\|g;\eta\rangle\rangle_{sc} \;=\; \frac{1}{\sqrt{2}}\Big(\|g;\eta\rangle\rangle_{sa} \;+\; \|\sigma_3\, g\, \sigma_3;\eta\rangle\rangle_{sa}\Big)\,. \tag{5.3}$$

Note that the brane associated to g is identical to the one associated to $\sigma_3 g \sigma_3$, and thus the resulting brane moduli space is $\mathrm{SU}(2)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ changes the sign of the off-diagonal entries of g.[4] Expressed in terms of the parameters of (2.6), conjugation by σ_3 simply corresponds to the shift $\psi \mapsto \psi + \pi$, but does not affect θ (nor ϕ). The radius perturbation, on the other hand, only affects θ, and thus the RG flow is compatible with the $\tilde{\mathcal{S}}$ orbifold. Alternatively, we can think about how the disc correlation functions (from which the bulk boundary coefficient that appears in the RG equation can be deduced) behave under the orbifold: since the currents in question live in a sector that is invariant under the orbifold action, the result is unchanged, and thus the old analysis applies. We can therefore conclude that

$$\|\left(\begin{smallmatrix} 0 & ie^{i\psi} \\ ie^{-i\psi} & 0 \end{smallmatrix}\right);\eta\rangle\rangle_{sc} \quad \overset{\delta R<0}{\Longleftarrow} \quad \|g;\eta\rangle\rangle_{sc} \quad \overset{\delta R>0}{\Longrightarrow} \quad \|\left(\begin{smallmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{smallmatrix}\right);\eta\rangle\rangle_{sc}\,. \tag{5.4}$$

Thus, as before, the superconformal branes of the circle theory flow to Dirichlet or Neumann branes as the radius is increased or decreased, respectively. There is however now a new subtlety: the end-point of the RG flow is not necessarily a fundamental brane in the circle theory. For example, for $\eta = -$, the resulting diagonal group element (to which the system flows if the radius is increased) describes a superposition of a BPS Dirichlet brane and anti-brane at opposite points on the circle, while for $\eta = +$ the diagonal group element describes a superposition of two non-BPS Dirichlet branes at opposite points on the circle. Unlike the situation in the superaffine orbifold, these branes are not fundamental in the circle theory.

[3]From the point of view of the superaffine theory, all of these branes are however fundamental and cannot be further resolved.

[4]It is therefore different from the $g \equiv -g$ equivalence in SO(3).

5.3 The circle theory at $R = \frac{M}{N}$

Finally, we want to comment on the situation where the radius of the superconformal circle theory is rational. As suggested in [2] (see also [23]) in the context of the bosonic analysis, we can make use of the fact that the circle theory at radius $R = \frac{M}{N}$ can be obtained as a $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifold of the circle theory at radius $R = 1$. In fact, the relevant orbifold action can be taken to be

$$\mathcal{S}_N \;:=\; \Big(X \;\;\mapsto\;\; X \;+\; \frac{2\pi R}{N}\Big)\,, \qquad \mathcal{W}_M \;:=\; \Big(\tilde{X} \;\;\mapsto\;\; \tilde{X} \;+\; \frac{\pi}{RM}\Big)\,. \tag{5.5}$$

The branes of the fractional radius theory can then be obtained from the branes at $R = 1$ in the usual manner. The only subtlety involves the determination of the fixed points. One finds that fixed points only appear if N is even: the non-BPS branes ($\eta = +$) of the $R = 1$ theory are fixed points under $\mathcal{S}_N^{N/2}$. The resolution of these fixed points then leads to the inclusion of a R-R component in the boundary state, and thus the branes with $\eta = +$ are BPS for N even. (On the other hand, the R-R part of the BPS branes at $R = 1$ is projected out for N even, and hence the $\eta = -$ branes are non-BPS.) Thus for even N the roles of the BPS and non-BPS branes are interchanged, in perfect agreement with what was already found (by some different reasoning) in [4].

Just like the $\tilde{\mathcal{S}}$ orbifold action in section 5.2, the orbifold operators $\mathcal{S}_N$ and $\mathcal{W}_M$ act on the boundary states as $g \mapsto \Gamma_M g \Gamma_M^{-1}$ and $g \mapsto \Gamma_N g \Gamma_N$, respectively, where $\Gamma_L = \mathrm{diag}(e^{\frac{i\pi}{L}}, e^{-\frac{i\pi}{L}})$ is defined as in [4]. In particular, these operators therefore only act on the phases ϕ, ψ in (2.6), but not on the modulus angle θ. (They also leave the sector in which the perturbing field lives invariant.) Thus as before the radius changing orbifold does not affect the RG flow analysis, and the result goes through directly. In general, though, the end-point of the RG flow will now be a superposition of a number of Dirichlet or Neumann branes.

6 Conclusions

In this paper we have studied the behaviour of the $\mathcal{N} = 1$ superconformal boundary conditions of the free boson and free fermion theory at $c = \frac{3}{2}$ under the radius-changing bulk deformation. The results are similar to those that were previously obtained in the bosonic case in [2]: if the radius is increased, a generic brane flows to a superposition of Dirichlet branes, while the endpoint of a radius decreasing perturbation is a superposition of Neumann branes.

As in the bosonic example, our analysis hinged on relating the circle theory to an SU(2) WZW model for which the RG flow can be solved explicitly. In the present context, the WZW model in question appears at level $k = 2$, and it is equivalent to the superaffine theory, rather than the circle theory directly. In addition, there were some subtleties involving the GSO projection.

Acknowledgements

This research has been partially supported by the Swiss National Science Foundation and the Marie Curie network 'Constituents, Fundamental Forces and Symmetries of the

Universe' (MRTN-CT-2004-005104). This paper is largely based on the Diploma thesis of O.S.

A Comparison of overlaps

In order to identify the boundary states of the superaffine theory with the WZW model it is useful to compare their overlaps. We begin with the analysis of the superaffine boundary states given in (4.2). Using the same techniques as in [4] one finds that their overlap equals

$$
\begin{aligned}
{}_{sa}\langle\langle g_1; - \| \, q^{L_0 - \frac{c}{24}} \, \| g_2; - \rangle\rangle_{sa} &= \frac{1}{2} \sum_{n\in\mathbb{Z}} \Big(f_3(\tilde{q}) \; + \; (-1)^n \, f_4(\tilde{q}) \Big) \, \frac{\tilde{q}^{\frac{1}{2}\left(-\frac{\alpha}{\pi}+n\right)^2}}{\eta(\tilde{q})} \\
{}_{sa}\langle\langle g_1; + \| \, q^{L_0 - \frac{c}{24}} \, \| g_2; + \rangle\rangle_{sa} &= f_3(\tilde{q}) \sum_{n\in\mathbb{Z}} \frac{\tilde{q}^{\frac{1}{2}\left(-\frac{\alpha}{\pi}+n\right)^2}}{\eta(\tilde{q})} \; ,
\end{aligned}
$$

where $\tilde{q}$ is the variable in the open string channel, and

$$
\cos(\alpha) \;=\; \frac{1}{2} \, \mathrm{Tr}\Big(g_1^{-1} \, g_2 \Big) \; . \tag{A.1}
$$

Here we have used the standard f_i functions from [20] that are defined as

$$
\begin{aligned}
f_2(q) \;=\; \sqrt{2} \, q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n) \qquad f_3(q) \;&=\; q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 + q^{n+\frac{1}{2}}) \\
f_4(q) \;&=\; q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 - q^{n+\frac{1}{2}}) \; .
\end{aligned} \tag{A.2}
$$

The calculation of the overlap between boundary states corresponding to different values of η depends on our convention concerning the relative normalisation of the NS-NS Ishibashi states; the convention we have used is that [5]

$$
{}^{\mathrm{NS}}\langle\langle j; m, n; \eta | \, q^{L_0 - \frac{c}{24}} \, | j; m, n; -\eta \rangle\rangle^{\mathrm{NS}} \;=\; (-1)^j \, f_4(q) \, \frac{q^{\frac{j^2}{2}} \; + \; q^{\frac{(j+1)^2}{2}}}{\eta(q)} \; . \tag{A.3}
$$

Then one finds that

$$
{}_{sa}\langle\langle g_1; - \| \, q^{L_0 - \frac{c}{24}} \, \| g_2; + \rangle\rangle_{sa} \;=\; \frac{f_2(\tilde{q})}{\sqrt{2}} \sum_{n \in \mathbb{Z} - \frac{1}{2}} \frac{\tilde{q}^{\frac{1}{2}\left(-\frac{\alpha}{\pi}+n\right)^2}}{\eta(\tilde{q})} \; . \tag{A.4}
$$

[5]This differs from the conventions of [9] by a factor of $(-1)^m$.

To compare to the usual level 2 WZW characters χ_j, we take $g_1 = \pm g_2$. Then one obtains

$$\begin{aligned}
{}_{sa}\langle\langle g;-\| \, q^{L_0-\frac{c}{24}} \, \| g;-\rangle\rangle_{sa} &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\Big(f_3(\tilde{q}) + \ (-1)^n \, f_4(\tilde{q})\Big) \, \frac{\tilde{q}^{\frac{1}{2}n^2}}{\eta(\tilde{q})} \ = \ \chi_{j=0}(\tilde{q}) \\
{}_{sa}\langle\langle g;-\| \, q^{L_0-\frac{c}{24}} \, \| -g;-\rangle\rangle_{sa} &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\Big(f_3(\tilde{q}) - \ (-1)^n \, f_4(\tilde{q})\Big) \, \frac{\tilde{q}^{\frac{1}{2}n^2}}{\eta(\tilde{q})} \ = \ \chi_{j=1}(\tilde{q}) \\
{}_{sa}\langle\langle g;+\| \, q^{L_0-\frac{c}{24}} \, \| g;+\rangle\rangle_{sa} &= \sum_{n\in\mathbb{Z}} f_3(\tilde{q}) \, \frac{\tilde{q}^{\frac{1}{2}n^2}}{\eta(\tilde{q})} \ = \ \chi_{j=0}(\tilde{q}) \ + \ \chi_{j=1}(\tilde{q}) \\
{}_{sa}\langle\langle g;-\| \, q^{L_0-\frac{c}{24}} \, \| g;+\rangle\rangle_{sa} &= \sum_{n\in\mathbb{Z}-\frac{1}{2}} \frac{f_2(\tilde{q})}{\sqrt{2}} \, \frac{\tilde{q}^{\frac{1}{2}n^2}}{\eta(\tilde{q})} \ = \ \chi_{j=\frac{1}{2}}(\tilde{q}) \ ,
\end{aligned} \tag{A.5}$$

where we have used standard theta-function identities (see for example [24]) to relate the expressions to the characters of the WZW model given in (3.4). The expressions on the right-hand side precisely agree with what one expects, based on the fusion rules of the su(2) level 2 theory. It is also not difficult to see how both sides generalise for general g_1 and g_2.

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